C*-algebras, right correspondences, and Q-systems II: C^*Alg is Q-system complete

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Q-system completion for C-2-Categories*

The Ohio State University

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Introduction and Motivation

- ♣ Q-systems are a unitary version of a Frobenius algebra object in a C*-2-category/unitary tensor category (UTC).
- ♡ Introduced in [Lon94] to characterize canonical endomorphism associated to a finite index subfactor of an infinite factor.
- ♠ Q-systems in a UTC give an axiomatization of the standard invariant of a finite index subfactor [Müg03].
- ♢ Subfactor reconstruction: all irreducible finite index extensions of II_1-factor are *crossed products* $N \subset N \rtimes_H Q$, where $Q \in \mathsf{C}$ is an indecomposable Q-system, and $H : \mathsf{C} \to \mathrm{Bim}(N)$ is a *UTC-action*.

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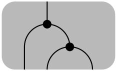
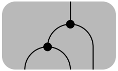
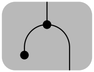
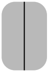
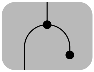
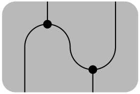
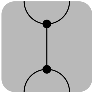
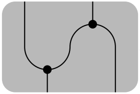
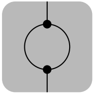
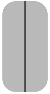
A **Q-system** in C is a 1-morphism $Q \in C(b \to b)$ with multiplication $m =$  and unit $i =$  satisfying:

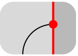
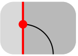
(Q1) Associativity:  $=$ ,

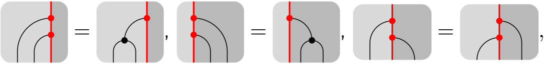
(Q2) Unitality:  $=$  $=$ ,

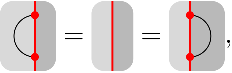
(Q3) Frobenius:  $=$  $=$ ,

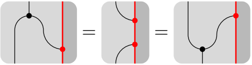
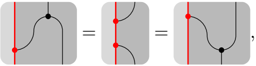
(Q4) Separable:  $=$ .

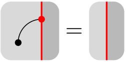
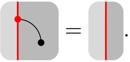
A bimodule $X \in \mathrm{C}(a \to b)$ over Q-systems $P \in \mathrm{C}(a \to a)$ and $Q \in \mathrm{C}(b \to b)$ consists of left and right actions

$\lambda =$  and $\rho =$ , satisfying

(B1) (associativity) 

(B2) (separable) 

(B3) (Frobenius)  and 

(B4) (unital)  and 

♣ (B3) and (B4) automatically follow from (B1) and(B2). [BKLR15]

Bimodule intertwiners

Given Q-systems $P \in C(a \to a)$ and $Q \in C(b \to b)$, and $P - Q$ bimodules $X \in C(a \to b)$ and $Y \in C(a \to b)$, we define

$$\mathrm{QSys}(C)({}_P X_Q \Rightarrow {}_P Y_Q)$$

to consists of all those $f \in C({}_a X_b \Rightarrow {}_a Y_b)$ such that

♣ This defines a C*-2-category QSys(C) with canonical embedding $\iota_C : C \to \mathrm{QSys}(C)$, mapping $C \ni c \mapsto 1_c$, the trivial Q-system; i.e the monoidal unit ${}_Q Q_Q \in C(Q \to Q)$.

▶ C is **Q-system complete**, iff ι_C defines a †-2-equivalence.

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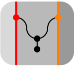
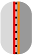
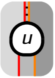
Composition of 1-morphisms

To compose the $P - Q$ bimodule ${}_A X_B$ and the $Q - R$ bimodule ${}_B Y_C$, we unitarily split the separability projector

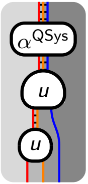
$p_{X,Y} :=$  $:=$  $=$  $= u_{X,Y}^{\dagger} \circ u_{X,Y}$

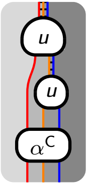
for a coisometry $u_{X,Y}$, unique up to unique unitary.

 $= X \otimes_Q Y$ 

$= u_{X,Y}.$

As in [NY16, Rem. 2.6], associator $\alpha^{\mathsf{QSys(C)}}$ uniquely determined by



$=$ 

$: (X \otimes Y) \otimes Z \to X \otimes_Q (Y \otimes_R Z).$

C*Alg : Right C*-correspondences

Specialize to C = C*Alg, the C*-2-category consisting of

- **0-mor:** Unital C*-algebras: A, B, C, ...
- **1-mor:** Right C*-Correspondences:
  ${}_AX_B \in \mathrm{C^*Alg}(A \to B), {}_BY_C \in \mathrm{C^*Alg}(B \to C), \ldots$
  A $\mathbb{C}$-vector space X with commuting left A- and right B-actions, and a right B-valued positive definite inner product:
  $$\langle \cdot | \cdot \rangle_B : \overline{X} \times X \to B.$$
  A left A-action on X by *adjointable operators*: A right B-linear map $T : X_B \to Z_B$ between right B-modules is *adjointable* if there is a right B-linear map $T^\dagger : Z_B \to X_B$ such that
  $$\langle \eta | T\xi \rangle_B = \langle T^\dagger \eta | \xi \rangle_B \qquad \forall \xi \in X, \ \forall \eta \in Z.$$
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Theorem: [CHPJP21]

$\mathrm{C^*Alg}$ is Q-system complete; i.e. $\mathrm{C^*Alg} \cong \mathrm{QSys}(\mathrm{C^*Alg})$.

Realization $|\cdot| : \mathrm{QSys}(\mathrm{C^*Alg}) \to \mathrm{C^*Alg}$ is inverse †-2-functor to $\iota_{\mathrm{C^*Alg}} : \mathrm{C^*Alg} \to \mathrm{QSys}(\mathrm{C^*Alg})$, is defined as follows:

♠ A Q-system $Q \in \mathrm{C^*Alg}(B \to B)$ maps to $|Q| := \mathrm{Hom}_{\mathbb{C}-B}(B \to Q)$:

$q_1 \cdot q_2 :=$

, $1_{|Q|} :=$
, $q^* :=$

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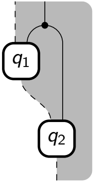
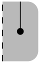
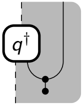
$q_1 \cdot q_2 :=$  , $1_{|Q|} :=$  , $q^* :=$  .

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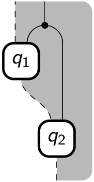
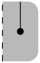
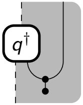
$q_1 \cdot q_2 :=$ 

, $1_{|Q|} :=$ 
, $q^* :=$ 

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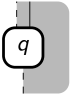
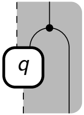
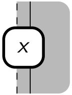
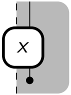


$\mapsto$ 



$\mapsto$ 

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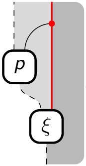
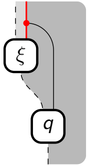
$p \rhd \xi :=$ 

, $\xi \lhd q :=$ 

$\forall f \in |P|$,
$\forall \eta \in |M|$, and
$\forall g \in |Q|$.

♠ $f \in \mathrm{C^*Alg}({}_AX_B \Rightarrow {}_AY_B)$ P-Q intertwiner maps to

$|f| : |X| \to |Y|$ given by $|f|\Big($

$\Big) :=$

$\in |Y|$.

$|f|$ is $|P|$-$|Q|$ bimodular.

▶ Unitarily splitting separability projectors $p_{X,Y} = u^\dagger_{X,Y} \circ u_{X_Y}$ gives the tensor structure for $|\cdot|$, and the splitting of $1_{|Q|}$, yielding the desired equivalence.

Realization of Q-systems...

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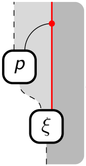
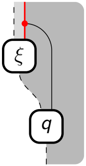
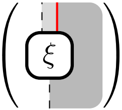
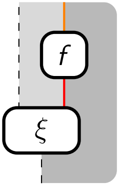
$p \rhd \xi :=$ 

, $\xi \lhd q :=$ 

$\forall f \in |P|$,
$\forall \eta \in |M|$, and
$\forall g \in |Q|$.

♠ $f \in \mathrm{C^*Alg}({}_AX_B \Rightarrow {}_AY_B)$ P-Q intertwiner maps to

$|f| : |X| \to |Y|$ given by $|f|$ 

$:=$ 

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$|f|$ is $|P|$-$|Q|$ bimodular.

▶ Unitarily splitting separability projectors $p_{X,Y} = u^\dagger_{X,Y} \circ u_{X_Y}$ gives the tensor structure for $|\cdot|$, and the splitting of $1_{|Q|}$, yielding the desired equivalence.

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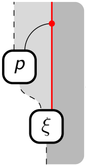
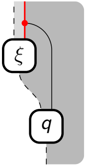
$p \triangleright \xi :=$ 

, $\xi \triangleleft q :=$ 

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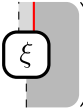
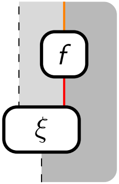
$|f| : |X| \to |Y|$ given by $|f|$ (

) $:=$ 

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Conclusions and Perspectives

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♠ Realization splits the problem of classifying finite index extensions of a II_1-factor N in two parts:

- (*Analytical:*) Constructing and classifying UTC-actions $H : \mathsf{C} \to \mathsf{Bim}(N)$.
  Generalization of classification of groups actions on II_1 factors, little is known for UTC.
- (*Algebraic:*) Classifying Q-systems in a UTC.
  Non-abelian cohomology problem.
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♢ Q-System completion induces new actions of UTCs Morita equivalent to C on finite extensions of N.

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